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CONCERNING SIMPLE CONTINUED FRACTIONS.

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1. In his *Lecons sur la théorie des fonctions*, Chapter II, Borel gives an account of Liouville's demonstration of the existence of non-algebraic numbers. As an application of the theorem on which this demonstration rests he considers the approximation of incommensurable numbers by the method of simple continued fractions and is led to an important proposition in the elementary theory of the subject. As his treatment of this proposition is different from that adopted in any standard text on Algebra—in fact the proposition in the general form is not is not in any Algebra with which I am acquainted—it is proposed to give with some modification of arrangement and detail Borel's account of the theorem and its application to continued fractions.

2. Denote by ξ a real number satisfying an irreducible equation $f(x)=0$ with integral coefficients, of degree n in x. Let p/q be a rational fraction in its lowest terms and let both p/q and ξ lie in a certain interval α, β, where α and β are both finite but otherwise arbitrary. Then the following proposition may be established:

THEOREM. *A positive number M can be determined such that whatever be the number p/q in the interval α, β the inequality*

$$(1)\quad \left|\frac{p}{q}-\xi\right|>\frac{1}{Mq^n}$$

is satisfied.

Since $f(x)$ is a polynomial in x it is finite and continuous in the interval

α, β and has a derivative which is also a polynomial, and consequently finite in the same interval. Hence there is a positive number M such that

$$(2) \qquad |f'(x)| < M$$

in the entire interval α, β.

Inasmuch as $f(x)$ is a polynomial with integral coefficients,

$$f(p/q) = A/q^n,$$

where A is an integer. If p/q is not a root of the equation $f(x)=0$ then $|A| \geqq 1$, and consequently

$$(3) \qquad \left| f\left(\frac{p}{q}\right) \right| \geqq \frac{1}{q^n}.$$

Now by Taylor's formula, since $f(\xi)=0$,

$$f(p/q) = (p/q - \xi) f'[\xi + \theta(p/q - \xi)], \; 0 < \theta < 1.$$

Since $\xi + \theta(p/q - \xi)$ lies in the interval α, β we may apply (2). Hence

$$|f(p/q)| < |p/q - \xi| \, M.$$

Then by inequality (3) we have (1) after a slight reduction.

3. Since ξ is a real number it may be expressed in the form of a continued fraction, in the usual notation

$$\xi = (a_0, a_1, \ldots, a_m, \xi_{m+1}),$$

where $\xi_{m+1} \geqq 1$, while $a_1, a_2, \ldots, a_m$ are positive integers, not zero. Denote the $(i+1)$th convergent of this continued fraction by p_i/q_i. Then

$$\xi = \frac{p_m \xi_{m+1} + p_{m-1}}{q_m \xi_{m+1} + q_{m-1}}$$

and

$$\frac{p_m}{q^m} - \xi = -\frac{p_{m-1} q_m - p_m q_{m-1}}{q_m^2 \left(\xi_{m+1} + \dfrac{q_{m-1}}{q_m}\right)} = \frac{(-1)^{m+1}}{q_m^2 \left(\xi_{m+1} + \dfrac{q_{m-1}}{q_m}\right)}.$$

Since $a_m > 0$, $q_{m-1}/q_m > 0$, and, consequently,

$$(4) \qquad \left| \frac{p_m}{q_m} - \xi \right| < \frac{1}{\xi_{m+1} \, q_m^2}.$$

Replacing p and q in inequality (1) by p_m and q_m respectively, and comparing the results with inequality (4) we have

$$(5) \quad \xi_{m+1} < M q_m^{n-2}.$$

Hence the inequality, as given by Borel,

$$(6) \quad a_{m+1} < M q_m^{n-2}.$$

4. When $n=1$, ξ is rational and inequality (6) takes the form

$$a_{m+1} < M/q_m.$$

Since q_m increases without limit with m, by taking m great enough a_{m+1} may be made less than any assigned positive number, however small. Now this involves a contradiction since always $a_{m+1} \geqq 1$. Hence the elementary

THEOREM. *A rational number is represented by a terminating continued fraction.*

5. When $n=2$, the inequality (6) becomes

$$(7) \quad a_{m+1} < M.$$

To determine M explicitly let

$$f(x) \equiv ax^2 + bx + c, \quad b^2 - 4ac \equiv D > 0,$$

and let ξ represent either of the values $(-b \pm \sqrt{D})/a$. Since

$$\left| \frac{p_m}{q_m} - \xi \right| < 1, \qquad m = 0, 1, 2, \ldots$$

two numbers α, β can be chosen in the interval $\xi - 1$, $\xi + 1$ so that ξ and every convergent p_m/q_m, $m=0, 1, 2, \ldots$, shall lie in the interval α, β. Hence when for M the greater of the values $|f'(\xi-1)|$, $|f'(\xi+1)|$ is taken, then in the interval α, β, $|f(x)| < M$. We find that

$$M = 2(|a| + \sqrt{D}).$$

Comparing this with inequality (7) we have the following

THEOREM. *In the continued fraction representing either quadratic irrationality $(-b \pm \sqrt{D})/a$, a, b, D integers, $D > 0$, every partial denominator after the first is less than* $2(|a| + \sqrt{D})$.

6. Denote by ε a small positive number. Then α and β may be chosen in the interval $\xi - \varepsilon$, $\xi + \epsilon$ so that for i great enough ξ and every convergent of order greater than i lies in the interval α, β. As in the preceding instance take for M the greater of the two numbers $|f'(\xi - \epsilon)|$, $|f'(\xi + \epsilon)|$. Then in the interval α, β,

$$|f'(x)| < M, \quad M = 2(|a| \varepsilon + \sqrt{D}).$$

Now let $2\sqrt{D}+d$ be the integer next greater than $2\sqrt{D}$ and choose ε so that

$$\varepsilon<\frac{d}{2\,|\,a\,|}, \text{ whence } M<2\sqrt{D}+d.$$

Hence by inequality (7), since a_{m+1} is an integer,

$$a_{m+1}<2\sqrt{D}$$

for every m, $m>i$. This is the well known

THEOREM. *In the continued fraction representing either quadratic irrationality* $(-b\pm\sqrt{D})/a$, a, b, D *integers*, $D>0$, *every partial denominator from one of a certain rank on, is less than* $2\sqrt{D}$.

7. The more general theorem of which this is a special case is the following

THEOREM. *In the continued fraction representing the real number* ξ, *where* ξ *is the root of an irreducible equation with integral coefficients* $f(x)=0$ *of degree* n, *every partial denominator* a_{m+1}, *from one of a certain rank on, satisfies the inequality* $a_{m+1}<\,|\,f'(\xi)\,|\,q_m^{\,n-2}$.

PROPERTIES OF THE FUNCTION $(1+a)^x$.

By ANTONIO LLANO, Scranton, Pa.

The following demonstrations of some well known theorems are submitted as being simpler and more systematic than those usually given. The binomial $1+a$ is supposed positive, or $a>-1$.

THEOREM I. *If* $x>1$, *then* $(1+a)^x>1+ax$.

Let $x=u/v$, where $u>v$, and put $1+a=z^v$. We have

$$\frac{z^u-1}{z^v-1}=\frac{z^{u-1}+z^{u-2}+\dots\dots+1}{z^{v-1}+z^{v-2}+\dots\dots+1}=1+\frac{z^v+z^{v+1}+\dots\dots+z^{u-1}}{z^{v-1}+z^{v-2}+\dots\dots+1}\dots\dots(1).$$

If $z>1$, the fraction in the final member is less than $\dfrac{(u-v)z^v}{vz^{v-1}}$.

$$\therefore\ \frac{z^u-1}{z^v-1}>1+\frac{(u-v)z}{v}>1+\frac{u-v}{v}, \text{ namely, } \frac{u}{v}\dots\dots(2).$$

$$\therefore z^u>1+(z^v-1)\frac{u}{v};\ \text{ or, } (1+a)^{u/v}>1+a\frac{u}{v}.$$

If $z<1$, the final member of (1) is less than $1+(u-v)z/v$, and the character of the inequalities in (2) is reversed; but, as z^v-1 negative, the inequalities